Tropical varieties

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In this measure point we present two different definitions of tropical varieties and (partly) show they are related. In the first definition we define tropical varieties combinatorially using t-initial ideals which is helpful when one want to do computations. The other definition uses the valuation map on the *Puisex series*. This definition highlight the connection between tropical varieties and ordinary affine varieties in algebraic geometry.

As a bonus fact we may mention that the name "*tropical*" was invented by french mathematicians in honor to the Brazilian mathematician Imre Simon who pioneered the field - tropical must be the french view of Brazil.

1 Prerequisites

Fix a field K then a valuation on K is a function: $val : K \to \mathbb{R} \cup \{\infty\}$ satisfying:

(1) $val(a) = \infty$ if and only if $a = 0$

(2) $val(ab) = val(a) + val(b)$

(3) $val(a + b) \geq \min\{val(a), val(b)\}$

We state and prove a little lemma we will need later

Lemma 1. *Let $a, b \in K$. If $val(a) \neq val(b)$ then $val(a + b) = \min\{val(a), val(b)\}$*

Proof. We may WLOG assume $val(b) > val(a)$. First observe by the second property of the valuation map

$$val(1) = val(1 \cdot 1) = val(1) + val(1) = 2val(1) \Rightarrow val(1) = 0$$

Also since $(-1)^2 = 1$ we get $val(-1) = 0$. This imply $val(-b) = val((-1)b) = val(-1) + val(b) = val(b)$. We now use the third property of the valuation map

$$val(a) = val((a + b) - b) \geq \min\{val(a + b), val(-b)\} = \min\{val(a + b), val(b)\}$$

But $val(a) < val(b)$ hence $val(a) \geq val(a + b)$. Now

$$val(a + b) = \min\{val(a), val(b)\} = val(a)$$

which proves the result. $\square$

The *Puiseux series* is the set:

$$\mathbb{C}\{\{t\}\} = \left\{ \sum_{k=m}^{\infty} a_k t^{\frac{k}{N}} \mid m \in \mathbb{Z},\, N \in \mathbb{N},\, a_k \in \mathbb{C} \right\}$$

One can show that this set is algebraically closed [theorem 2.1.4, [1]]. This field come with a valuation $val : \mathbb{C}\{\{t\}\} \to \mathbb{Q} \cup \{\infty\}$ by taking a series to the exponent of the first term i.e. if $\sum_{k=m}^{\infty} a_k t^{\frac{k}{N}} \in \mathbb{C}\{\{t\}\}$ then $val\left(\sum_{k=m}^{\infty} a_k t^{\frac{k}{N}}\right) = \min\{\frac{k}{N} \,|a_k \neq 0\}$. We define $val(0) = \infty$.

2 Definition of a Topical variety

In this section we define tropical varieties combinatorially using initial ideals. Consider a polynomial f in the polynomial ring $\mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$. The terms in f are of the form $ct^a x^v$ with $c \in \mathbb{C}^*$, $a \in \mathbb{Q}$ and $v \in \mathbb{N} \cup \{0\}$. We define a degree on the terms

Definition 1. For $w \in \mathbb{R}^n$ the t-w-degree of a term $ct^a x^n$ is defined as $-val(ct^a) + \langle w, v\rangle = -a + \langle w, v\rangle$. Furthermore for $f \in \mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$ non zero, we define the t-initial form w.r.t w as the sum of terms in f with maximal t-w-degree and with t evaluated in 1. We denote this by t-$in_w(f)$. We have t-$in_w(f) \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ see remark 1.

Example 1. *Let* $w = (-2, -1)$ *and* $f = (2t + t^{\frac{4}{2}} + t^{\frac{3}{2}})x^2 + (-3t^3 + 2t^4)y^2 + t^5 xy^2 + (t + 3t^2)x^7 y^2 \in \mathbb{C}\{\{t\}\}[x, y]$ *then one can compute that the maximal* t-w-*degree is -5 and hence* t-$in_w(f) = 2x^2 - 3y^2$

Remark 1. *One can ask if this really makes sense? We must be sure that the maximum is really attained and that we do not pick out an infinite amount of terms. Since the exponents of* t *in a Puisex series are bounded from below and we only have finite many variables* x_i *the maximum is attained at least ones and only a finitely many times. That is why we can consider* t-$in_w(f)$ *as an element in* $\in \mathbb{C}[x_1, x_2, \ldots, x_n]$ *for any* $f \in \mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$ *non zero. If* $g, f \in \mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$ *non zero then*

$$t\text{-}in_w(fg) = t\text{-}in_w(f)t\text{-}in_w(g) \tag{1}$$

We are now ready to define what the t-initial ideal is

Definition 2. Let $I \subset \mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$ be an ideal and $w \in \mathbb{R}^n$. The t-initial ideal of I w.r.t w is

$$t\text{-}in_w(I) = \langle t\text{-}in_w(f) \mid f \in I \backslash \{0\}\rangle \subset \mathbb{C}[x_1, x_2, \ldots, x_n]$$

Now one could get the idea that if $G \subset \mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$ is a generating set for an ideal $I \subset \mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$ then the initial forms of the elements in G would also generate the t-initial ideal of I, but this is not correct in general as the next example shows:

Example 2. *Let* $I = \langle tx + y, x + t\rangle \subset \mathbb{C}\{\{t\}\}[x, y]$ *and pick* $w = (1, -1)$. *Notice*

$$t\text{-}in_w(tx + y) = tx_{|t=1} = x, \quad t\text{-}in_w(x + t) = x$$

and hence $\langle t\text{-}in_w(tx + y), t\text{-}in_w(x + t)\rangle = \langle x\rangle$. *But* $y - t^2 = tx + y - t(x + t) \in I$ *and clearly* $y = t\text{-}in_w(y - t^2) \notin \langle x\rangle$ *hence* $t\text{-}in_w(I) \neq \langle t\text{-}in_w(tx + y), t\text{-}in_w(x + t)\rangle$.

We can now state the definition of a tropical variety:

Definition 3. Let $I \subset \mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$ then the tropical variety of I is

$$\mathcal{T}(I) := \{w \in \mathbb{R}^n \mid \ t\text{-}in_w(I) \text{ does not contain a monomial}\}$$

We present a lemma to be used later

Lemma 2. *Let* $I \subset \mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$ *be an ideal and* $w \in \mathbb{R}^n$. *Then* $w \in \mathcal{T}(I)$ *if and only if* t-$in_w(f)$ *is not a monomial for all* $f \in I \backslash \{0\}$.

Proof. We show first "⇒": Assume $w \in \mathcal{T}(I)$ which means t-$in_w(I)$ does not contain a monomial. Since t-$in_w(f) \in t$-$in_w(I)$ for all $f \in I \backslash \{0\}$ it is clear that for all $f \in I \backslash \{0\}$ we have that t-$in_w(f)$ is not a monomial. Next we prove "⇐": We show the contrapositive. Assume $w \notin \mathcal{T}(I)$. Then by definition of $\mathcal{T}(I)$ we know there exists a monomial x^α in t-$in_w(I)$. That is, there exists $f_i \in I$ and $g_i \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ such that $x^\alpha = \sum_i g_i t\text{-}in_w(f_i)$. By collecting terms we may find h_i's such that $x^\alpha = \sum_j t\text{-}in_w(h_j)$. Let W be the maximum t-w-degree running over all h_j. We may find $\beta_j, N_j \in \mathbb{Z}$ such that $t^{\frac{\beta_j}{N_j}} h_j$ max'l t-w-degree equals W. From remark (1) and the definition of the t-initial form we have

$$t\text{-}in_w(t^{\frac{\beta_j}{N_j}} h_j) = t\text{-}in_w(t^{\frac{\beta_j}{N_j}})t\text{-}in_w(h_j) = t\text{-}in_w(h_j)$$

because we substitute t with 1. This imply $\sum_j t\text{-}in_w(h_j) = \sum_j t\text{-}in_w(t^{\frac{\beta_j}{N_j}} h_j)$. Now all the terms coming from the $t\text{-}in_w(t^{\frac{\beta_j}{N_j}} h_j)$'s has the same t-w-degree which means that the terms w picks out in $\sum_j t\text{-}in_w(t^{\frac{\beta_j}{N_j}} h_j)$ are exactly the same as w picks out in $t\text{-}in_w(\sum_j t^{\frac{\beta_j}{N_j}} h_j)$ i.e.

$$x^\alpha = \sum_j t\text{-}in_w(h_j) = \sum_j t\text{-}in_w(t^{\frac{\beta_j}{N_j}} h_j) = t\text{-}in_w(\sum_j t^{\frac{\beta_j}{N_j}} h_j)$$

Since $\sum_j t^{\frac{\beta_j}{N_j}} h_j \in I$ we have found a polynomial f in I such that $t\text{-}in_w(f)$ is a monomial. $\square$

3 Another definition of a tropical variety

In this section we shortly show how one can also define the tropical variety of an ideal.

Consider the valuation map *val* on $\mathbb{C}\{\{t\}\}$. This can be extended to give at map

$$Val : (\mathbb{C}\{\{t\}\})^n \to (\mathbb{Q} \cup \{\infty\})^n$$

where Val is the coordinatewise valuation *val*.

Definition 4. Let $I \subset \{\{t\}\}[x_1, x_2, \ldots, x_n]$ be an ideal then this defines a (normal) variety $V(I) \subset \mathbb{C}\{\{t\}\}$. The tropical variety $trop(I)$ of I is defined as

$$\begin{aligned} trop(I) &:= -\overline{Val(V(I)) \cap \mathbb{Q}^n} \\ &= -\overline{\{(val(u_1), val(u_2), \ldots, val(u_n)) \mid (u_1, u_2, \ldots, u_n) \in V(I)\} \cap \mathbb{Q}^n} \end{aligned}$$

where we take the usual topological closure in $\mathbb{R}^n$

4 The fundamental theorem and properties of tropical varieties

The first definition of a tropical variety do not give any insight in the relation between tropical varieties and algebraic varieties while the second definition do just that. Conversely the first definition is more helpful if one wish to do computations. In this section we will (partly) show that the two definitions are equivalent.

First we present some properties of tropical varieties. First of all the union of two tropical varieties is a tropical variety.

Proposition 1. *Let $I, J \subset \mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$ be ideals. Then $\mathcal{T}(I) \cup \mathcal{T}(J) = \mathcal{T}(I \cap J)$.*

Proof. We show the two inclusions. Assume $w \notin \mathcal{T}(I \cap J)$ i.e. $t\text{-}in_w(I \cap J)$ contain a monomial. Because $I \cap J \subset I$ and $I \cap J \subset J$ we have $t\text{-}in_w(I \cap J) \subset t\text{-}in_w(I)$ and $t\text{-}in_w(I \cap J) \subset t\text{-}in_w(J)$. That is, both $t\text{-}in_w(I)$ and $t\text{-}in_w(J)$ contain a monomial hence $w \notin \mathcal{T}(I) \cup \mathcal{T}(J)$. Now suppose $w \notin \mathcal{T}(I) \cup \mathcal{T}(J)$. Then both $\mathcal{T}(I)$ and $\mathcal{T}(J)$ contain a monomial. By Lemma 2 there exists $f \in I$ and $g \in J$ such that $t\text{-}in_w(f)$ and $t\text{-}in_w(g)$ are monomials. By identity (1) we have $t\text{-}in_w(fg) = t\text{-}in_w(f)t\text{-}in_w(g)$ which is a monomial and since both I and J are ideals we also have $fg \in I \cap J$. That is, $\mathcal{T}(I \cap J)$ contain a monomial i.e. $w \notin \mathcal{T}(I \cap J)$. $\square$

Using induction one can show that this is also true if we consider $I_1, I_2, \ldots, I_k$ ideals in $\mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$.

Let $I \subset R$ be an ideal in some commutative ring. Recall the definitions of its radical:

$$\sqrt{I} = \{x \in R \,|\, x^n \in I \quad \text{for some positive integer } n\}$$

We will show that the tropical variety does not change under the radical of the defining ideal.

Proposition 2. *Let $I \subset \mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_m]$ be an ideal. Then $\mathcal{T}(I) = \mathcal{T}(\sqrt{I})$.*

Proof. Since $I \subset \sqrt{I}$ clearly by the definition of the tropical variety we have $\mathcal{T}(I) \supset \mathcal{T}(\sqrt{I})$. Assume $w \notin \mathcal{T}(\sqrt{I})$ then by lemma 2 there exist $f \in \sqrt{I}$ and $n \in \mathbb{Z}$, $n > 0$ (the radical of an ideal is an ideal) such that $t\text{-}in_w(f)$ is a monomial and $f^n \in I$. By applying identity 1 several times we get $t\text{-}in_w(f)^n = t\text{-}in_w(f^n)$. That is we have found a monomial in $\mathcal{T}(I)$ w.r.t w hence $w \notin \mathcal{T}(I)$. $\square$

We now bring the fundamental theorem. The theorem stated as below can be widely more generalized see theorem 3.2.4 in [1].

Theorem 1. *Let $I \subset \mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$ be an ideal. Then*

$$\mathbb{Q}^n \cap \mathcal{T}(I) = -Val\left(V(I) \cap (\mathbb{C}\{\{t\}\}^*)^n\right)$$

We give a proof of the inclusion $\supseteq$.

Proof. Let $p = (p_1, p_2, \ldots, p_n) \in V(I) \cap (\mathbb{C}\{\{t\}\})^n$. Then $p_i \in \mathbb{C}\{\{t\}\}^*$, $val(p_i) < \infty$ and $-Val(p) \in \mathbb{Q}^n$. We then need to show that $t\text{-}in_{-Val(p)}(I)$ do not contain a monomial. By lemma 2 we only need to show that $t\text{-}in_{-Val(p)}(f)$ is not a monomial for all $f \in I \backslash \{0\}$.

Assume $f \in I \backslash \{0\}$. Then $f(p) = 0$ because $p \in V(I)$. Write $f(p) = \sum_u c_u t^\alpha p^u$(where $p^u = p_1^{u_1} \cdot p_2^{u_2} \cdots p_n^{u_n}$. Since $f(p) \in \mathbb{C}\{\{t\}\}$ we can apply the valuation: $val(f(p)) = val(0) = \infty > val(c_u t^\alpha p^u)$ because $p \neq 0$. This implies that the maximum of

$$-val(c_u t^\alpha p^u) = -val(c_u t^\alpha) - u_1 \cdot val(p_1) - u_2 \cdot val(p_2) - \cdots - u_n \cdot val(p_n) = -\alpha + \langle -Val(p), u \rangle$$

is attained at least twice. This means that $-Val(p)$ picks out at least two terms in f hence $t\text{-}in_{-Val(p)}(f)$ is not a monomial. $\square$

For the other inclusion one can in the case of a zero dimensional ideal I actually give a constructive proof, and by the proper restrictions one can for example use *Singular* to compute a point in $V(I)$ for a giving point in the tropical variety $\mathcal{T}(I)$ see [2].

5 Computational aspects

In this section we are interested in computing t-initial ideals and tropical varieties.

Lemma 3. *Let $f \in \mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$. For at principal ideal $\langle f \rangle \subset \mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$ we have*

$$\mathcal{T}(\langle f \rangle) = \{w \in \mathbb{R}^n \mid \ t\text{-}in_w(f) \text{ is not a monomial}\}$$

Proof. Let $w \in \mathcal{T}(\langle f \rangle) = \{w \in \mathbb{R}^n \mid t\text{-}in_w(\langle f \rangle)$ does not contain a monomial$\}$. Observe $t\text{-}in_w(f) \in t\text{-}in_w(\langle f \rangle)$ hence $w \in \{w \in \mathbb{R}^n \mid \ t\text{-}in_w(f)$ is not a monomial$\}$. Let
$w \in \{w \in \mathbb{R}^n \mid \ t\text{-}in_w(f)$ is not a monomial $\}$. Using lemma 2 we need to check that all t-initial forms for elements from $\langle f \rangle$ are not monomials. Such an element is of the form gf with $g \in \mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$. From identity (1) we have $t\text{-}in_w(gf) = t\text{-}in_w(g) t\text{-}in_w(f)$ and since $t\text{-}in_w(f)$ it not a monomial, $t\text{-}in_w(gf)$ is also not a monomial. $\square$

Example 3. *Let $I = \langle x + y + 1 \rangle \subset \mathbb{C}\{\{t\}\}[x, y]$. Then by the above lemma*

$$\mathcal{T}(I) = \{w \in \mathbb{R}^n \mid t\text{-}in_w(x + y + 1) \text{ is not a monomial}\}$$

Write $w = (w_1, w_2)$. Since $val(x) = val(y) = val(1) = 0$ we have that $t\text{-}in_w(x + y + 1)$ is not a monomial exactly when $w_1 = w_2 \geq 0$ or $w_1 = 0 \geq w_2$ or $w_2 = 0 \geq w_1$. Then the tropical variety is:

$$\mathcal{T}(I) = \{(w_1, w_2) \mid w_1 = w_2 \geq 0\} \cup \{(w_1, w_2), \mid w_1 = 0 \geq w_2\} \cup \{(w_1, w_2) \mid w_2 = 0 \geq w_1\} \qquad (2)$$

See figure 0.1. The tropical variety defined by a principal ideals is called a tropical hypersurface.

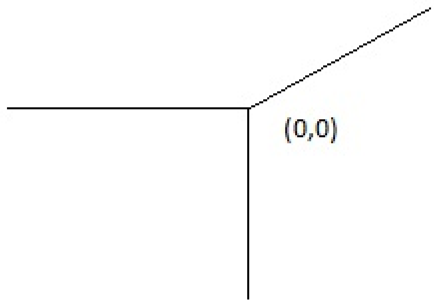

Figure 0.1: The tropical variety from example 3

For the ideal in example 3 we can actually see directly that theorem 1 is true. Let $I = \langle x+y+1\rangle \subset \mathbb{C}\{\{t\}\}[x,y]$. Observe

$$V(I)\cap(\mathbb{C}\{\{t\}\}^*)^2 = V(x+y+1)\cap(\mathbb{C}\{\{t\}\}^*)^2 = \{(u,-u-1)\,|\,u\in\mathbb{C}\{\{t\}\},\quad u\neq 0,-1\}$$

From identity 2 we see that $t\text{-}in_w(f)$ is not a monomial if and only if w is a positive multiple of either $(1,1)$, $(0,-1)$ or $(-1,0)$. Notice also that for $(u,-1-u)\in V(I)\cap(\mathbb{C}\{\{t\}\}^*)^2$ we have

$$(val(u), val(-1-u)) = \begin{cases} (val(u), val(u)) & \text{if } val(u) < 0 \\ (val(u), 0) & \text{if } val(u) > 0 \\ (0,\alpha) & \text{if } u = -1 + ct^\alpha + z, \text{ where } z \text{ has higher evaluation than } u \\ (0,0) & \text{otherwise} \end{cases}$$

these computations rests heavily on lemma 1. For example if $val(u) > 0$ then $val(-u) = val(u) > 0$ and lemma 1 imply $val(-1-u) = val((-1)+(-u)) = \min\{val(-1), val(-u)\} = \min\{0, val(u)\} = 0$. Now it is clear that if u is an element in $V(I)\cap(\mathbb{C}\{\{t\}\})^2$ then $-Val((u,-u-1)) = (-val(u), -val(-1-u))$ is in $\mathbb{Q}^2\cap\mathcal{T}(I)$. Conversely if $w = (w_1,w_2)\in\mathbb{Q}\times\mathbb{Q}$ pick out a non monomial $t\text{-}in_w(x+y+1)$ then w is a positive multiple of $(1,1)$, $(0,-1)$ or $(-1,0)$. We clearly can find $u\in\mathbb{C}\{\{t\}\}^*$ such that $-Val((u,-1-u)) = (w_1,w_2)$ i.e. $w\in -Val\left(V(I)\cap(\mathbb{C}\{\{t\}\}^*)^2\right)$

Next we ask the question: "how can we compute the t-initial ideal over the polynomial ring $\mathbb{C}[t,x_1,x_2,\ldots,x_n]$?" We would like to use algorithms we already know.

Lemma 4. *Let $I\subset\mathbb{C}[t,x_1,x_2,\ldots,x_n]$ be an ideal. Then $t\text{-}in_w(I) = \left(in_{(-1,w)}(I)\right)_{|t=1}$ where $in_w(I)$ is defined in definition 1.6.1 in [4].*

Proof. We clearly have $t\text{-}in_w(f) = \left(t\text{-}in_{(-1,w)}(f)\right)_{|t=1}$ for $f\in I\backslash\{0\}$. Then

$$\begin{aligned} t\text{-}in_w(I) &= \langle t\text{-}in_w(f)\,|\,f\in I\backslash\{0\}\rangle \\ &= \langle (t\text{-}in_{(-1,w)}(f))_{|t=1}\,|\,f\in I\backslash\{0\}\rangle \\ &= (t\text{-}in_(-1,w)(I))_{|t=1} \end{aligned}$$

□

Since we know how to compute $in_{(-1,w)}(I)$ we can also compute $t\text{-}in_w(I)$. When working with ideals over the polynomial rings $\mathbb{Q}[x_1,x_2,\ldots,x_n]$ or $\mathbb{C}[x_1,x_2,\ldots,x_n]$ one can also do as follows:

Definition 5. Let $I\subset\mathbb{Q}[x_1,x_2,\ldots,x_n]$ (or in $\mathbb{C}[x_1,x_2,\ldots,x_n]$). Then the tropical variety if I is defined by

$$\mathcal{T}(I) = \{w\in\mathbb{R}^n\,|\,in_w(I) \text{ does not contain a monomial}\}$$

In many situations it suffices only to consider tropical varieties defined as above see section 6.3 in [3]. We state without proof:

Proposition 3. *Let $I \subset \mathbb{C}[t, x_1, x_2, \ldots, x_n]$ be an ideal and put $J = \langle I \rangle_{\mathbb{C}\{\{t\}\}[x_1,x_2,\ldots,x_n]}$ i.e. the ideal generated by I when we may multiply arbitrary elements from $\mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$. Then for all $w \in \mathbb{R}^n$ we have $t\text{-}in_w(I) = t\text{-}in_w(J)$.*

Notice $\mathbb{C} \subset \mathbb{C}\{\{t\}\}$ hence a polynomial in $\mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$ could also happen to be in $\mathbb{C}[x_1, x_2, \ldots, x_n]$, when this occur we say that the polynomial has constant coefficients. The following lemma justify the above definition of tropical varieties for constant coefficient polynomials.

Lemma 5. *Let $I \subset \mathbb{C}[x_1, x_2, \ldots, x_n]$ be an ideal and $J = \langle I \rangle_{\mathbb{C}\{\{t\}\}[x_1,x_2,\ldots,x_n]} \subset \mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_n]$. Then for all $w \in \mathbb{R}^n$ we have $in_w(I) = t\text{-}in_w(J)$.*

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